The Matrix Formulation of Scattering Problems

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Abstract—Two regions in space are coupled through an opening in a perfectly conducting surface. By using a complete set of eigenvectors in the opening, each region can be represented by an equivalent Norton circuit involving a short-circuit current (a vector) and a generator admittance (a matrix). The particular case of a cavity at resonance is investigated. Application to a cavity terminated in a waveguide is considered, and the transformation of the equivalent circuit resulting from the shift of the terminal plane is analyzed. After solving the example of a slotted waveguide, a possible set of eigenvectors for an arbitrary opening is proposed.

I. Introduction

A TYPICAL "coupled regions" configuration is shown in Fig. 1, where a field $\bar{e}_i$, $\bar{h}_i$ is incident on a metallic cavity I bounded by an infinitely thin conducting wall S. The wall is provided with an aperture S'. The fields in Regions I and II can be computed (in principle at least) once the tangential component $\bar{e}_{\text{tang}}$ of the electric field in S' is known. Suitable assumptions can be made concerning $\bar{e}_{\text{tang}}$ in certain particular cases (e.g., for small holes and for slots). In general, however, $\bar{e}_{\text{tang}}$ must be determined by:[1]

1) expressing $\bar{h}_{\text{tang}}$ on the cavity side in terms of $\bar{e}_{\text{tang}}$
2) expressing $\bar{h}_{\text{tang}}$ in Region II in terms of $\bar{e}_{\text{tang}}$, and
3) equating the two values of $\bar{h}_{\text{tang}}$ in S', and solving the resulting integral equation for $\bar{e}_{\text{tang}}$.

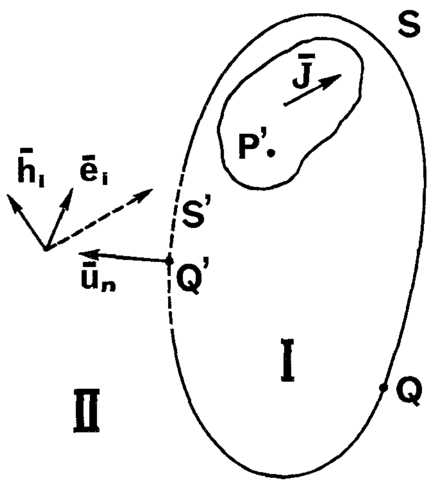

Fig. 1. A typical "coupled regions" problem.

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[1] For the solution of an actual problem, see F. J. Kriegler, F. E. Mills, and J. Van Bladel, "Fields excited by periodic beam currents in a cavity-loaded tube," *J. Appl. Phys.*, vol. 35, pp. 1721–6, June 1964.

In this paper, we seek to formulate the problem in terms of an equivalent network problem. Truly, the computational work is not simplified by this approach, but we believe that some conceptual clarity can be achieved by showing the connection between the electromagnetic problem and the (perhaps) more familiar network structure. Our treatment remains very general, and we leave for future reports the application of the network formulation to cavity filters, periodic structures, etc. Basically, our method rests on the use of an old workhorse—the eigenvector method—to the determination of the fields in the aperture.

II. Scattering of an Incident Field by a Cavity with an Opening

The cavity shown in Fig. 1 is excited by the volume currents $\bar{J}$ and by the aperture fields in S'. We shall first assume that the frequency does not coincide with one of its resonant values, leaving for Section III a discussion of the phenomena at resonance. Under those circumstances, the electromagnetic field in the cavity is uniquely determined by the values of $\bar{J}$ and $\bar{u}_n \times \bar{E}$ on S'. The two contributions are additive. For an evacuated cavity, for example, the tangential magnetic field due to $\bar{J}$ is[2]

$$\bar{H}_g{}^{\mathrm{I}}(Q) = \iiint \left[\sum_m \frac{k_m \bar{h}_m(Q)\bar{e}_m(P')}{(k_m{}^2 - k^2)} \right] \cdot \bar{J}(P') dV' \quad (1)$$

where k_m is one of the resonant wave numbers, and $\bar{e}_m$ and $\bar{h}_m$ are the normalized solenoidal eigenvectors. These are connected by the relationships

$$\bar{h}_m = \frac{1}{k_m} \operatorname{curl} \bar{e}_m \quad \text{and} \quad \bar{e}_m = \frac{1}{k_m} \operatorname{curl} \bar{h}_m.$$

Clearly, $\bar{H}_g{}^{\mathrm{I}}$ is the field which exists on S' when the latter surface is short circuited. The contribution from $(\bar{u}_n \times \bar{E})$ can be written as

$$\begin{aligned}\bar{H}_b{}^{\mathrm{I}}(Q) &= \iint_S \Bigg[-\frac{1}{j\omega\mu} \sum \bar{g}_m(Q)\bar{g}_m(Q') \\ &\qquad - j\omega\epsilon \sum \frac{\bar{h}_m(Q)\bar{h}_m(Q')}{k_m{}^2 - k^2} \Bigg] \cdot \bar{u}_n \times \bar{E}(Q') dS' \\ &= \iint_{S'} \mathcal{G}^{\mathrm{I}}(Q \mid Q') \cdot \bar{u}_n \times \bar{E}(Q') dS' \quad (2)\end{aligned}$$

[2] J. Van Bladel, *Electromagnetic Fields*. New York: McGraw-Hill, 1964, pp. 299, 415, 500, and 504.

where $\bar{g}_m$ denotes a normalized irrotational magnetic eigenvector.

The field $\bar{H}_{\text{tang}}$ on the outer side of S' is similarly given by the sum of a "short-circuit" component $\bar{H}_g^{II}(Q)$ and a contribution due to $(-\bar{u}_n)\times\bar{E}$ (we write $-\bar{u}_n$ because it is the unit vector along the *outer* normal of Region II which should be used). This contribution can be written as

$$\bar{H}_b^{II}(Q) = -\iint_{S'} \mathcal{G}^{II}(Q \mid Q')\cdot\bar{u}_n \times \bar{E}(Q')dS'. \tag{3}$$

Equating the two values of $\bar{H}_{\text{tang}}$ on both sides of S' leads to the integral equation

$$\iint_{S} [\mathcal{G}^{I}(Q \mid Q') + \mathcal{G}^{II}(Q \mid Q')]\cdot\bar{u}_n \times \bar{E}(Q')dS' = \bar{H}_g^{II}(Q) - \bar{H}_g^{I}(Q). \tag{4}$$

At this point we introduce[3] a set of vectors $\bar{\alpha}_m$, complete in S', and satisfying the orthonormality property

$$\iint_{S'} \bar{\alpha}_m\cdot\bar{\alpha}_k{}^* dS' = \delta_{mk}. \tag{5}$$

Utilizing the expansions

$$\bar{E} = \sum_m V_m\bar{\alpha}_m$$

$$\bar{J}_s{}^{I} = \bar{H}_g{}^{I}\times\bar{u}_n = \sum_m I_{gm}{}^{I}\bar{\alpha}_m$$

$$\bar{J}_s{}^{II} = \bar{H}_g{}^{II}\times(-\bar{u}_n) = \sum_m I_{gm}{}^{II}\bar{\alpha}_m$$

$$\mathcal{G}^{I}(Q \mid Q') = -\sum_n\sum_p Y_{np}{}^{I}\bar{u}_n\times\bar{\alpha}_n(Q)\bar{u}_n\times\bar{\alpha}_p{}^*(Q')$$

$$\mathcal{G}^{II}(Q \mid Q') = -\sum_n\sum_p Y_{np}{}^{II}(-\bar{u}_n\times\bar{\alpha}_n(Q)) \cdot(-\bar{u}_n\times\bar{\alpha}_p{}^*(Q')) \tag{6}$$

and equating the values of $\bar{H}\times\bar{u}_n$ on both sides of S' leads to the following network equations

$$I_{g1}{}^{I} - Y_{11}{}^{I}V_1 - Y_{12}{}^{I}V_2 \cdots = -(I_{g1}{}^{II} - Y_{11}{}^{II}V_1 - Y_{12}{}^{II}V_2 - \cdots)$$

$$I_{g2}{}^{I} - Y_{21}{}^{I}V_1 - Y_{22}{}^{I}V_2 \cdots = -(I_{g2}{}^{II} - Y_{21}{}^{II}V_1 - Y_{22}{}^{II}V_2 - \cdots) \tag{7}$$

etc. $\cdots$.

which can be written more concisely as

$$\bar{I}_g{}^{II} + \bar{I}_g{}^{I} = (\mathcal{Y}^{I} + \mathcal{Y}^{II})\cdot\bar{V}. \tag{8}$$

It will be noticed that $\bar{J}_S{}^{I}$ and $\bar{J}_S{}^{II}$ are the surface current densities on the short-circuited surface S'. A matrix such as $\mathcal{Y}^{I}$ has a simple physical interpretation. From (7), indeed, it is the linear relationship which exists between the tangential electric field on S' and the resulting field $\bar{u}_n\times\bar{H}_b$ on the cavity side of S'. Thus,

"Projection $\bar{u}_n\times\bar{H}_b{}^{I}$ on the $\bar{\alpha}$ space
$=\mathcal{Y}^{I}\times$projection of $\bar{E}$ on the $\bar{\alpha}$ space."

In circuit terms, $\mathcal{Y}^{I}$ is the admittance looking into I, i.e., the ratio between $\bar{H}\times\bar{u}_n$, and $\bar{E}_{\text{tang}}$, where $\bar{u}_n$ is the unit vector pointing inside I. The sign conventions embodied in (6) ensure that the admittance looking into a matched waveguide load, for instance, is equal to the *positive* characteristic resistance R_c (and not to $-R_c$). This point will be belabored in Section IV.

The circuit equation (8) can be represented schematically as in Fig. 2. It is seen that the field problem reduces to the determination of $\bar{V}$, i.e., of the inverse of the $\mathcal{Y}^{I}+\mathcal{Y}^{II}$ matrix. The $\mathcal{Y}$ matrix has the usual properties of an admittance matrix. Assume, for example, that the cavity contains an anisotropic medium whose ϵ and μ are Hermitian tensors ($\tilde{\epsilon}=\epsilon^*$ and $\tilde{\mu}=\mu^*$). It is easy to

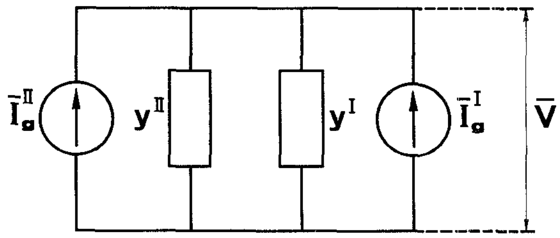

Fig. 2. Equivalent network for a scattering problem.

show, by classical methods,[2,4] that

$$Y_{ik} = -Y_{ki}{}^* \tag{9}$$

i.e., that the admittance matrix is skew-Hermitian. For symmetric tensors and real base vectors $\bar{\alpha}$, the matrix becomes symmetric, i.e.,

$$Y_{ik} = Y_{ki}. \tag{10}$$

III. Cavities at Resonance, Coupled Cavities, and Quadrupoles

Assume for simplicity that Cavity I does not contain any volume sources and is filled with a nondissipative medium. When k coincides with one of the resonant values k_ν, the Green's dyadic becomes infinite. The fields in the aperture, however, must remain finite because of the finite value of Q of the loaded cavity. It is therefore necessary [see (2)] that the tangential field satisfy the relationship.

$$\lim_{k\to k_\nu}\iint_{S'} (\bar{u}_n\times\bar{E})\cdot\bar{h}_\nu dS = 0. \tag{11}$$

[3] The idea of expanding the tangential fields in a complete set seems to be due to A. Tonning, "On the network description of electromagnetic field problems," Rept. AFCRL-62-967, 1962. Tonning utilizes two biorthogonal sets, one for $\bar{E}_{\text{tang}}$ and one for $\bar{H}_{\text{tang}}$.

[4] See, e.g., Tonning,[3] or R. F. Harrington and A. T. Villeneuve, "Reciprocity relationships for gyrotropic media," *IEEE Trans. on Microwave Theory and Techniques*, vol. MTT-6, pp. 308–310, July 1958.

The situation is not unlike that of Fig. 3, where the voltage across LC becomes zero at resonance (while the current remains finite) even though the admittance becomes infinite. For the cavity, similarly, the coefficient of excitation of the resonant mode $\bar{e}_\nu$, $\bar{h}_\nu$ must remain finite. It follows that the expression

$$\lim_{k \to k_\nu} \left[\iint_{S'} \frac{(\bar{u}_n \times \bar{E}) \cdot \bar{h}_\nu dS}{k_\nu^2 - k^2} \right] \tag{12}$$

must remain finite. To further examine this limit, notice that the real eigenvector $\bar{h}_m$ can be represented by the following expansion, valid in S':

$$\bar{h}_m(Q') = \sum_n B_{nm} \bar{u}_n \times \bar{\alpha}_n(Q') = \sum_n B_{nm}^* \bar{u}_n \times \bar{\alpha}_n^*(Q').$$

It is then a simple matter to show that each resonant mode contributes a term

$$\frac{j\omega\epsilon}{k_\nu^2 - k^2} B_{m\nu} B_{n\nu}^*$$

to Y_{mn}. Let $\mathcal{Y}'$ be the matrix obtained by deleting the contribution of mode ν from $\mathcal{Y}^{\text{I}}$. The circuit equations can now be written in the following way to emphasize the contribution of mode ν:

$$I_{g1}^{\text{II}} = (Y_{11}' + Y_{11}^{\text{II}})V_1 + (Y_{12}' + Y_{12}^{\text{II}})V_2 + \cdots + \frac{j\omega\epsilon}{k^2 - k_\nu^2} B_{1\nu}[B_{1\nu}^* V_1 + B_{2\nu}^* V_2 + \cdots]$$

$$I_{g2}^{\text{II}} = (Y_{21}' + Y_{21}^{\text{II}})V_1 + (Y_{22}' + Y_{22}^{\text{II}})V_2 + \cdots + \frac{j\omega\epsilon}{k^2 - k_\nu^2} B_{2\nu}[B_{1\nu}^* V_1 + B_{2\nu}^* V_2 + \cdots]. \tag{13}$$

etc. $\cdots$.

As k approaches k_ν, the quantity $D_\nu = B_{1\nu}^* V_1 + B_{2\nu}^* V_2 + \cdots$ must approach zero in such a manner that

$$\lim_{k \to k_\nu} j\omega\epsilon(B_{1\nu}^* V_1 + \cdots) = C(k^2 - k_\nu^2)$$

where C should be determined in order to fully evaluate the fields in the cavity. The formal solution proceeds by rewriting (13) in terms of C, and adding the condition $D_\nu = 0$ (the uncoupling condition) to the equations, viz.

$$(Y_{11}' + Y_{11}^{\text{II}})V_1 + (Y_{12}' + Y_{12}^{\text{II}})V_2 + \cdots + B_{1\nu} C = I_{g1}^{\text{II}}$$

$$(Y_{21}' + Y_{21}^{\text{II}})V_1 + (Y_{22}' + Y_{22}^{\text{II}})V_2 + \cdots + B_{2\nu} C = I_{g2}^{\text{II}}$$

$$B_{1\nu}^* V_1 + B_{2\nu}^* V_2 + \cdots = 0. \tag{14}$$

This is a system of equations with as many equations as unknowns, and out of which $V_1 V_2, \cdots, C$ can be determined.

The schematic diagram of Fig. 2 clearly shows that the region situated on a given side of the opening can be represented by a current generator in parallel with an admittance matrix. In this extension of Norton's theorem, generator and matrix have poles at the eventual resonant frequencies of the region. When two cavities are coupled together through a common aperture, the resonant frequencies of the total structure are determined by the condition that $(\mathcal{Y}^{\text{I}} + \mathcal{Y}^{\text{II}}) \cdot \bar{V} = 0$ admits a nonzero solution for $\bar{V}$. This condition implies that the determinant of the (infinite) system vanishes,[5] which is the desired equation for the eigenfrequencies.

The equivalent network method also can be used to analyze the composite structure shown in Fig. 4. We now introduce two sets of vectors, $\bar{\alpha}_m'$ and $\bar{\alpha}_m''$, respectively complete and orthonormal on S' and S''. The network equations take the form

$$\bar{I}_g^{\text{II}} + \bar{I}_g' = (\mathcal{Y}' + \mathcal{Y}^{\text{II}}) \cdot \bar{V}' + \mathcal{Y}^m \cdot \bar{V}''$$

$$\bar{I}_g^{\text{III}} + \bar{I}_g'' = \mathcal{Y}^p \cdot \bar{V}' + (\mathcal{Y}'' + \mathcal{Y}^{\text{III}}) \cdot \bar{V}''. \tag{15}$$

As before, a matrix such as $\mathcal{Y}'$ represents the linear relationship between $\bar{E}_{\text{tang}}$ and the resulting tangential field $\bar{\mu}_{n'} \times \bar{H}_b$ on S'. The currents $\bar{I}_g'$ and $\bar{I}_g''$ are the short-circuit currents produced by the volume sources of I on the short-circuited surfaces S' and S'', respectively. In the quadrupole equations (15), symmetry properties exist for the mutual admittance matrices $\mathcal{Y}^m$ and $\mathcal{Y}^p$. Remembering that $\mathcal{Y}^m \cdot \bar{V}''$, for example, represents the field, $\bar{\mu}_n' \times \bar{H}$ produced on the short-circuited surface S' by the electric field $\bar{E} = \Sigma V_m'' \bar{\alpha}_m''$ on S'', it is easy to show[4] that

$$Y_{12}^p = -(Y_{21}^m)^* \tag{16}$$

[5] For an early application of this method to the nosed-in klystron cavity, see W. C. Hahn, "A new method for the calculation of cavity resonators." *J. Appl. Phys.*, vol. 12, pp. 62–68, January 1941.

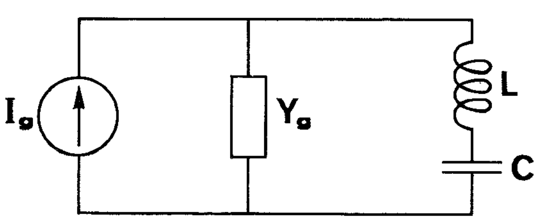

Fig. 3. A source connected to a resonant circuit.

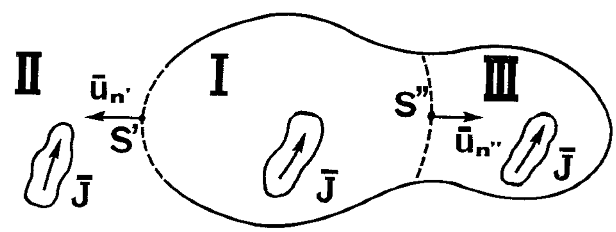

Fig. 4. Series combination of "coupled regions."

when the medium inside I is Hermitian, and

$$Y_{12}{}^{p} = Y_{21}{}^{m} \tag{17}$$

when the medium is symmetric and the $\bar{\alpha}$'s are real.

The quadrupole equations (15) allow one to apply the whole body of network theory to the cascade connection of electromagnetic structures. The structure to the left of S'' in Fig. 4, for example, can be replaced by a short-circuit current

$$\bar{I}_g'' - \mathcal{Y}^p\cdot(\mathcal{Y}' + \mathcal{Y}^{\mathrm{II}})^{-1}\cdot(\bar{I}_g{}^{\mathrm{II}} + \bar{I}_g') \tag{18}$$

in parallel with an admittance

$$\mathcal{Y}'' + \mathcal{Y}^p\cdot(\mathcal{Y}' + \mathcal{Y}^{\mathrm{II}})^{-1}\cdot\mathcal{Y}^m. \tag{19}$$

IV. Application to a Waveguide Problem

Figure 5 shows a cavity I terminated by a waveguide arm. Seen from the cross section S', the cavity has an (assumedly given) admittance matrix $\mathcal{Y}'$. It is our purpose to determine the admittance $\mathcal{Y}''$ in S''.

The eigenvectors $\bar{\alpha}_n$ suitable for the present problem are, from classical waveguide theory,[2]

$$\mathrm{grad}\ \phi_{mp}$$

$$\bar{u}_n \times \mathrm{grad}\ \Psi_{ns} = \bar{u}_z \times \mathrm{grad}\ \Psi_{ns}$$

where

$$\nabla_{xy}{}^2\Phi_{mp} + \mu_{mp}{}^2\Phi_{mp} = 0 \qquad \Phi_{mp} = 0 \text{ on contour } C \text{ of } S'$$

$$\nabla_{xy}{}^2\Psi_{ns} + \nu_{ns}{}^2\Psi_{ns} = 0 \qquad \frac{\partial\Psi_{ns}}{\partial n} = 0 \text{ on contour } C \text{ of } S'.$$

These eigenvectors are normalized in such a way that

$$\iint_S (\mathrm{grad}\ \Phi_{mp})^2 dS = \iint_S (\mathrm{grad}\ \Psi_{ns})^2 dS = 1.$$

The operator ∇^2 and the eigenvalues $\mu_{mp}{}^2$ and $\nu_{ns}{}^2$ are real. The eigenvectors can therefore be taken as real. We shall assume that all modes are damped except the lowest TE mode (subscript 11), and set

$$\gamma_{mp}{}^2 = \mu_{mp}{}^2 - k^2$$
$$\delta_{ns}{}^2 = \nu_{ns}{}^2 - k^2 \qquad (n, s \neq 1, 1)$$
$$k_{11}{}^2 = k^2 - \nu_{11}{}^2$$

with $k^2 = \omega^2\epsilon_0\mu_0$. The fields on S' can be expanded as

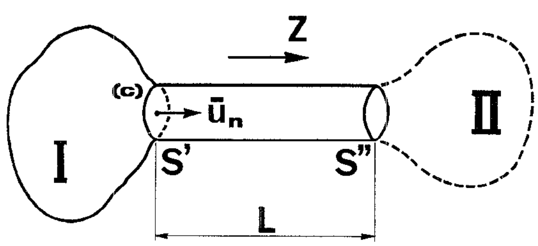

Fig. 5. Cavity with a waveguide output.

$$\bar{E} = \sum_{mp} V_{mp}' \,\mathrm{grad}\ \Phi_{mp} + \mathcal{V}_{11}' \,\mathrm{grad}\ \Psi_{11} \times \bar{u}_z + \sum_{n,s\neq 1,1} \mathcal{V}_{ns} \,\mathrm{grad}\ \Psi_{ns} \times \bar{u}_z$$

$$\bar{H} \times \bar{u}_z = \bar{H} \times \bar{u}_n = \sum_{mp} I_{mp}' \,\mathrm{grad}\ \Phi_{mp} + \mathcal{I}_{11}' \,\mathrm{grad}\ \Psi_{11} \times \bar{u}_z + \sum_{n,s\neq 1,1} \mathcal{I}_{ns}' \,\mathrm{grad}\ \Psi_{ns} \times \bar{u}_z. \tag{20}$$

Similar equations can be written for the fields on S'', provided the primes are replaced by double primes. Solution of the waveguide equations gives the following relationship between "voltages" and "currents." For the propagated mode:

$$\mathcal{V}_{11}' = \mathcal{V}_{11}'' \cos k_{11}L + jR_c\mathcal{I}_{11}'' \sin k_{11}L$$

$$\mathcal{I}_{11}' = \frac{j}{R_c}\mathcal{V}_1'' \sin k_{11}L + \mathcal{I}_{11}'' \cos k_{11}L \tag{21}$$

where $R_c = \omega\mu_0/R_{11}$. For the damped modes:

$$V' = V'' \cosh\gamma L + Z_c I'' \sinh\gamma L \tag{22}$$

$$I' = \frac{1}{Z_c} V'' \sinh\gamma L + I'' \cosh\gamma L$$

and

$$\mathcal{V}' = \gamma'' \cosh\delta L + \mathcal{Z}_c\mathcal{I}'' \sinh\delta L$$

$$\mathcal{I}' = \frac{1}{\mathcal{Z}_c}\mathcal{V}'' \sinh\delta L + \mathcal{I}'' \cosh\delta L \tag{23}$$

where

$$Z_c = \frac{\gamma}{j\omega\epsilon_0} \quad \text{and} \quad \mathcal{Z}_c = \frac{j\omega\mu_0}{\delta}\,.$$

The primed voltages and currents which appear in (21) through (23) are relative to the fields to the right of S'. The magnetic field to the left of S' is given by the vector

$$\bar{I}' = \bar{I}_g' - \mathcal{Y}'\cdot\bar{V}' \tag{24}$$

where $\bar{I}_g'$, for example, is the column vector representing the short-circuit field

$$\bar{H}_g' \times \bar{u}_n = \sum_{mp} u_{mp}' \,\mathrm{grad}\ \Phi_{mp} + \mathcal{U}_{11}' \,\mathrm{grad}\ \Psi_{11} \times \bar{u}_z + \sum_{n,s\neq 1,1} \mathcal{U}_{ns}' \,\mathrm{grad}\ \Psi_{ns} \times \bar{u}_z.$$

The components of $\bar{I}_g'$ are, therefore, u_{11}', u_{12}', $\cdots$, $\mathcal{U}_{11}'$, $\mathcal{U}_{12}'$, $\cdots$. Introducing the values of $\bar{I}'$ and $\bar{V}'$ obtained from (21) through (23) into (24) leads to an equation of the type

$$\bar{I}'' = \bar{I}_g'' - \mathcal{Y}''\cdot\bar{V}'' \tag{25}$$

where the desired Norton $\bar{I}_g''$ and $\mathcal{Y}''$ can easily be found after some cumbersome algebraic manipulation.

One obtains

$$\bar{I}_g'' = [C + \mathcal{Y}'\cdot\mathcal{Y}_c^{-1}\cdot S]^{-1}\cdot\bar{I}_g' \tag{26}$$

$$\mathcal{Y}'' = [C + \mathcal{Y}'\cdot\mathcal{Y}_c^{-1}\cdot S]^{-1}\cdot[\mathcal{Y}'\cdot C + \mathcal{Y}_c\cdot S] \tag{27}$$

where the diagonal matrices $\mathcal{Y}_c$, C, and S are given by

$$\mathcal{Y}_c = \begin{bmatrix} \frac{1}{Z_{c11}} & 0 & 0 & \cdots & & \\ 0 & \frac{1}{Z_{c12}} & & & & \\ & & \ddots & & & \\ 0 & & & \frac{1}{R_c} & & \\ 0 & & & & \frac{1}{Z_{c12}} & \\ \vdots & & & & & \ddots \end{bmatrix}$$

$$C = \begin{bmatrix} \cosh\gamma_{11}L & 0 & 0 & \cdots & \\ 0 & \cosh\gamma_{12}L & & & \\ & & \ddots & & \\ 0 & & \cos k_{11}L & & \\ \vdots & & & \cosh\delta_{12}L & \\ \vdots & & & & \ddots \end{bmatrix}$$

$$S = \begin{bmatrix} \sinh\gamma_{11}L & 0 & \cdots & \cdots & \\ 0 & \sinh\gamma_{12}L & & & \\ & & \ddots & & \\ 0 & & \sin k_{11}L & & \\ \vdots & & & \sinh\delta_{12}L & \\ \vdots & & & & \ddots \end{bmatrix}.$$

Equations (25) through (27) allow one to see how the Norton terms vary with the distance L. For very large L, the contribution from the damped modes tends to vanish, and (25) takes the form

$$\begin{pmatrix} I_{11}'' \\ I_{12}'' \\ \vdots \\ \mathfrak{I}_{11}'' \\ \mathfrak{I}_{12}'' \\ \vdots \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ \mathfrak{I}''_{g11} \\ 0 \\ \vdots \end{pmatrix} - \begin{pmatrix} \frac{1}{Z_{c11}} & 0 & \cdots & \\ 0 & \frac{1}{Z_{c12}} & & \\ & & \ddots & \\ 0 & & \mathcal{Y}_{11}'' & \\ \vdots & & & \frac{1}{Z_{c12}} \end{pmatrix} \cdot \begin{pmatrix} V_{11}'' \\ V_{12}'' \\ \vdots \\ \mathcal{V}_{11}'' \\ \mathcal{V}_{12}'' \\ \vdots \end{pmatrix} \tag{28}$$

where the contribution from the propagated mode has the value

$$\mathfrak{I}_{g11}'' = \frac{\mathfrak{I}_{g11}'}{\cos k_{11}L + jR_c\mathcal{Y}_{11}'\sin k_{11}L}$$

$$\mathcal{Y}_{11}'' = \frac{1}{R_c}\,\frac{R_c\mathcal{Y}_{11}' + j\tan k_{11}L}{1 + jR_c\mathcal{Y}_{11}'\tan k_{11}L}\,. \tag{29}$$

The actual value of the voltage vector $\bar{V}''$, i.e. of the coefficient of excitation of the various modes at the level of cross section S'', cannot be determined unless the input admittance of Volume II is given (Fig. 5). If II is an infinitely long waveguide, for instance, one has, in addition to (28), the relationship

$$\bar{I}'' = \mathcal{Y}_c\cdot\bar{V}''.$$

Comparison of the two equations leads to the expected result that all V's vanish except $\mathcal{V}_{11}''$, i.e., that no damped modes exist at the "junction" S''.

To illustrate the actual calculation of an equivalent circuit, we consider the slot antenna shown in Fig. 6, where the contour of the cross section is arbitrary. It is assumed that the slot is resonant, so that the voltage across the slot varies according to the relationship

$$v = V_0\cos\frac{\pi}{L}(Z - Z_0)$$

where Z_0 refers to the position of the center of the slot, and L (approximately equal to $\lambda/2$) is the length of the slot. Let c_0 be the value of the contour coordinate at the center of the slot. We assume that the voltage across the slot is a constant, independent of the loading conditions, so that the tangential electric field in the slot is of the form

$$\bar{E}_{\text{tang}} = V_0\cos\frac{\pi}{L}(Z - Z_0)\bar{u}_c\delta(c - c_0). \tag{30}$$

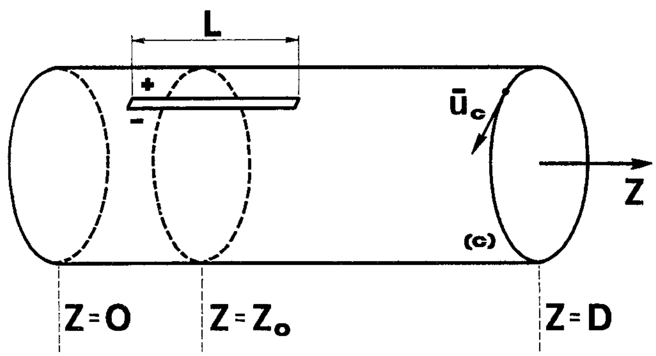

Fig. 6. Waveguide with a slot.

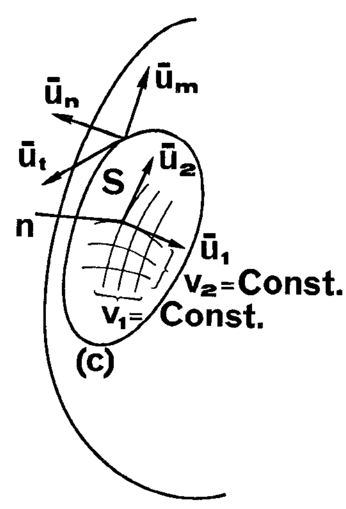

Fig. 7. Opening of arbitrary shape.

Utilizing this value of the boundary excitation in the waveguide equations[2] allows determination of the cross-sectional fields at $z'=D$, i.e., of $\mathfrak{J}_{g11}''$ and $\mathcal{Y}_{11}''$. With a short circuit at $z=0$, one finds, for example,

$$\overline{E} = \mathcal{V}_{11} \text{ grad } \psi_{11} \times \bar{u}_z$$

$$\overline{H} \times \bar{u}_z = \Bigg[\underbrace{\frac{2\pi \nu_{11}{}^2 \psi_{11}(c_0)}{j\omega\mu_0 L N_{11}{}^2 \left(k_{11}{}^2 - \dfrac{\pi^2}{L^2}\right)} \cos \frac{k_{11}L}{2} \frac{\sin (k_{11}Z_0)}{\sin (k_{11}D)} V_0}_{\mathfrak{J}_{g11}''} - \underbrace{\frac{k_{11}}{j\omega\mu_0 \tan k_{11}D}}_{\mathcal{Y}_{11}''} \mathcal{V}_{11}'' \Bigg] \text{grad } \Psi_{11} \times \bar{u}_z$$

in which the normalization factor

$$N_{11}{}^2 = \iint (\text{grad } \Psi_{11})^2 dS$$

has been left unspecified, and where $\psi_{11}(c_0)$ is the value of the eigenfunction at the location of the slot.

The choice of the most suitable eigenvectors $\bar{\alpha}$ is simple for a waveguide cross section. For an opening S of arbitrary shape, we propose the following generalization for the waveguide eigenvectors.[6] We choose orthogonal coordinates v_1, v_2, taken for instance along the lines of curvature of the surface[2], and such that the direction of increasing v_1, the direction of increasing v_2, and the positive normal n form a right-handed system of axes (see Fig. 7). An increase dv_1, dv_2 in the value of the coordinates results in a displacement dl of magnitude.

$$dl = (h_1{}^2 dv_1{}^2 + h_2{}^2 dv_2{}^2)^{1/2}.$$

Some important surface differential operators for a scalar function $f(v_1, v_2)$ are

$$\text{grad}_s f = \frac{1}{h_1} \frac{\partial f}{\partial v_1} \bar{u}_1 + \frac{1}{h_2} \frac{\partial f}{\partial v_2} \bar{u}_2$$

$$\nabla_S{}^2 f = \frac{1}{h_1 h_2} \frac{\partial}{\partial v_1}\left(\frac{h_2}{h_1} \frac{\partial f}{\partial v_1}\right) + \frac{1}{h_1 h_2} \frac{\partial}{\partial v_2}\left(\frac{h_1}{h_2} \frac{\partial f}{\partial v_2}\right)$$

where $\bar{u}_1$ and $\bar{u}_2$ are vectors of unit length and tangent, respectively, to the curves of constant v_2 and constant v_1, and directed to increasing coordinates. The Dirichlet functions of interest are now defined by the relationships

$$\nabla_S{}^2 \Phi_{mp} + \mu_{mp}{}^2 \Phi_{mp} = 0$$
$$\Phi_{mp} = 0 \text{ on } (C).$$

The Neumann eigenfunctions are defined by

$$\nabla_S{}^2 \Psi_{ns} + \nu_{ns}{}^2 \Psi_{ns} = 0$$
$$\frac{\partial \Psi_{ns}}{\partial m} = 0 \text{ on } (C)$$

where m is a direction in the tangent plane, perpendicular to (C), and directed outward from the region enclosed by (C). Utilizing the basic relationships[2]

$$\iint_S (A \nabla_S{}^2 B - B \nabla_S{}^2 A)\, dS = \int_C (A \text{ grad}_s B - B \text{ grad}_s A) \cdot \bar{u}_m\, dC \quad (31)$$

$$\iint_S (A \nabla_S{}^2 B + \text{grad}_s A \cdot \text{grad}_s B)\, dS = \int_C A(\bar{u}_m \cdot \text{grad } B)\, dS \quad (32)$$

it is easy to show that the eigenvalues μ^2 and ν^2 are real and non-negative (and hence, that the eigenfunctions can be chosen real), that the Φ's form an orthogonal system and the Ψ's another orthogonal system, and that the vectors

$$\text{grad}_s \Phi_{mp}$$
$$\text{grad}_s \Psi_{ns} \times \bar{u}_n$$

form an orthogonal system on S. These vectors are normal to the contour (C) of the aperture, and are therefore particularly well suited for expanding the vectors $\overline{E}_{\text{tang}}$ and $\overline{H} \times \bar{u}_n$, which are also normal to (C).

[6] An actual example of application of these generalized eigenvectors can be found in Van Bladel,[2] p. 467, where the aperture S' is a spherical cap.